

Applications of ordered abelian semi-groups and ordered semi-rings

Ulrich Oertel

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PRELIMINARY

Abstract

We describe a construction of ordered algebraic structures (ordered abelian semi-groups, ordered commutative semi-rings, etc.) and describe applications to topology, measure theory, and probability theory. For a suitable ordered semi- algebraic structure \mathbb{L} and measure space X we define \mathbb{L} -measures ν on X . In certain cases and for certain ordered structures \mathbb{L} , a measure ν can be interpreted as a “probability \mathbb{L} -measure” and can be used to calculate the probability of “black swan events.” If L is a codimension-1 essential lamination in a manifold, it sometimes admits transverse \mathbb{L} -measures for various \mathbb{L} . Transverse \mathbb{L} -measures can be used to understand laminations far more complicated than those admitting transverse \mathbb{R} -measures. We also investigate actions on \mathbb{L} -trees which are associated to laminations with transverse \mathbb{L} -measures.

1 Ordered algebraic structures.

In this paper we construct certain ordered semi- algebraic structures which have surely been described before. The author would appreciate any references to the literature. Then we use these ordered algebraic structures for various applications. In particular, we describe measures with values in the ordered semi- algebraic structures, some of which can be used for probability calculations. The (probability) measure theory ideas may also already exist in some form. Finally, we will describe transverse measures for codimension-1 laminations with values in our ordered semi- algebraic structures.

We begin by defining a very simple ordered commutative semi-ring which was used in [3] to analyze finite height (or finite depth) measured laminations in surfaces.

Example 1.1. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$ and let \mathbb{S} denote the set $\{0\} \cup (\mathbb{N}_0 \times (0, \infty])$, where $(0, \infty] \subset \bar{\mathbb{R}}$ and where $\bar{\mathbb{R}} = [-\infty, \infty]$ denotes the extended real line. We are using the symbol 0 to denote 0 in \mathbb{R} or \mathbb{Z} , but in the definition of \mathbb{S} we are adding a distinct 0. We use the lexicographical order relation on $\mathbb{N}_0 \times (0, \infty]$, so that $(i, t) < (j, s)$ either if $i < j$ and $s, t \in (0, \infty]$ or if $i = j$ and $t < s$. The element $0 \in \mathbb{S}$ is a least element, $0 < (i, t)$ for all

$(i, t) \in (\mathbb{N}_0 \times (0, \infty])$ We make \mathbb{S} a topological space with the order topology as shown in Figure 1.

The elements (i, ∞) are called *infinities* of \mathbb{S} . Define a commutative *addition* operation on \mathbb{S} by

$$(i, t) + (j, u) = \begin{cases} (i, t + u) & \text{if } i = j \\ (i, t) & \text{if } i > j \\ (j, u) & \text{if } j > i \end{cases}$$

$$(i, t) + 0 = (i, t).$$

with the convention $\infty + a = a + \infty = \infty$ for any $a \in (0, \infty]$. Define a commutative *multiplication* on \mathbb{S} by

$$(i, t)(j, u) = (i + j, tu),$$

$$0(i, t) = (i, t)0 = 0,$$

with the convention that $a\infty = \infty a = \infty$ for any $a \in (0, \infty]$.

We say $(i, t) \in (\mathbb{Z} \times (0, \infty])$ is *real* if $i = 0$. When we denote an element of \mathbb{S} by a single symbol $x = (i, t) \in \mathbb{S} \setminus \{0\}$, we will use $\mathfrak{L}(x) = i$ to denote the *level of* x and $\mathfrak{R}(x) = t$ to denote the *real part of* x , which lies in $(0, \infty]$. We make the convention that $\mathfrak{R}(0) = 0$ and $\mathfrak{L}(0)$ is undefined.

In terms of an operation we will define later, $\mathbb{S} = \mathbb{N}_0 \oslash [0, \infty]$.

It is routine to verify the following:

Lemma 1.2. *\mathbb{S} is an ordered commutative semi-ring with multiplicative identity $(0, 1)$ and additive identity 0 .*

We observe that the subset of \mathbb{S} which we identify with the positive extended reals, namely $\{(0, t) \in \mathbb{S} : t \in (0, \infty]\} = \{x \in \mathbb{S} : \mathfrak{L}(x) = 0\}$, has the usual addition and multiplication of the positive extended reals, and also has the usual topology.

To avoid confusion, we define the terms “ordered abelian semi-group,” “ordered commutative semi-ring” and “ordered semi-field” as used in this paper.

Definition 1.3. An *ordered abelian semi-group* is a set G equipped with a binary operation $+$, a total ordering $<$ and an element 0 satisfying the following axioms for any $a, b, c \in G$:

- (i) $(a + b) + c = a + (b + c)$
- (ii) $0 + a = a + 0 = a$
- (iii) $a + b = b + a$
- (iv) 0 is the least element and for all a and b , $a + b \geq b$.

Definition 1.4. An *ordered commutative semi-ring* is a totally ordered set R , with order $<$, equipped with two binary operations addition, $+$, and multiplication and with elements $0, 1$, such that $(R, +)$ is an ordered abelian semigroup and such that the following are satisfied for any $a, b, c \in R$:

- (i) $ab = ba$

- (ii) $(ab)c = a(bc)$
- (iii) $1a = a1 = a$
- (iv) $a(b + c) = (ab) + (ac)$
- (v) $0a = a0 = 0$

An *ordered semi-field* is a commutative semi-ring with the additional property that every non-zero element a has a multiplicative inverse \bar{a} such that:

- (ix) $a\bar{a} = \bar{a}a = 1$

Next we define an ordered commutative semi-ring \mathbb{O} , related to \mathbb{S} , which is pictured with the order topology in Figure 1.

Definition 1.5. Let \mathbb{O} denote the set $\{0\} \cup (\mathbb{Z} \times (0, \infty])$ where $(0, \infty] \subset \bar{\mathbb{R}}$ and where $\bar{\mathbb{R}} = [-\infty, \infty]$ denotes the extended real line. The definition of an ordering and operations on \mathbb{O} are essentially the same as for \mathbb{S} . We use the lexicographical order relation on $\mathbb{Z} \times (0, \infty]$, so that $(i, t) < (j, s)$ either if $i < j$ and $s, t \in (0, \infty]$ or if $i = j$ and $t < s$. The element $0 \in \mathbb{O}$ is a least element, $0 < (i, t)$ for all $(i, t) \in (\mathbb{Z} \times (0, \infty])$. We make \mathbb{O} a topological space with the order topology.

The elements (i, ∞) are called *infinities* of \mathbb{O} . Define a commutative *addition* operation on \mathbb{O} by

$$(i, t) + (j, u) = \begin{cases} (i, t + u) & \text{if } i = j \\ (i, t) & \text{if } i > j \end{cases}$$

$$(i, t) + 0 = (i, t),$$

with the convention $\infty + a = a + \infty = \infty$ for any $a \in (0, \infty]$. Define a commutative *multiplication* on \mathbb{O} by

$$(i, t)(j, u) = (i + j, tu),$$

$$0(i, t) = (i, t)0 = 0,$$

with the convention that $a\infty = \infty a = \infty$ for any $a \in (0, \infty]$.

We say $(i, t) \in (\mathbb{Z} \times (0, \infty])$ is *real* if $i = 0$. When we denote an element of \mathbb{O} by a single symbol $x = (i, t) \in \mathbb{O} \setminus \{0\}$, we will use $\mathfrak{L}(x) = i$ to denote the *level* of x and $\mathfrak{R}(x) = t$ to denote the *real part* of x , which lies in $(0, \infty]$. We make the convention that $\mathfrak{R}(0) = 0$ and $\mathfrak{L}(0)$ is undefined.

We define *vectors* in \mathbb{O}^n as n -tuples of elements of \mathbb{O} , and give \mathbb{O}^n the product topology. We allow n infinite, giving an infinite product. If $\lambda \in \mathbb{O}$ and $w \in \mathbb{O}^n$, $w = (w_1, \dots, w_n)$, then we define *scalar multiplication* by $\lambda w = (\lambda w_1, \lambda w_2, \dots, \lambda w_n)$. The *lattice points* of \mathbb{O}^n are vectors of the form $[(i_1, \infty), (i_2, \infty), \dots, (i_n, \infty)]$; we use Z to denote the set of lattice points in \mathbb{O}^n . The *origin* is the vector with all entries 0. A *cone* in \mathbb{O}^n is a subset of \mathbb{O}^n closed under scalar multiplication by $\lambda \in \mathbb{O}$.

We observe that the subset of \mathbb{O} which we identify with the positive extended reals, namely $\{(0, t) \in \mathbb{O} : t \in (0, \infty]\} = \{x \in \mathbb{O} : \mathfrak{L}(x) = 0\}$, has the usual addition and multiplication of the positive extended reals, and also has the usual topology. Scalar multiplication

shifts the levels of all entries of a vector by the same integer. Scalar multiplication by a real does not shift levels of entries.

Clearly there is an action of \mathbb{Z} on \mathbb{O}^n . Namely, for any integer r the action takes $w \in \mathbb{O}^n$, $w = (w_1, \dots, w_n)$ to the scalar product $(r, 1)(w_1, \dots, w_n)$, shifting the levels of all entries by r . The action on \mathbb{O} fixes 0, which means that subspaces of \mathbb{O}^n of the form $w_{i_1} = w_{i_2} = \dots = w_{i_s} = 0$ are preserved by the action. (We use notation for finite products, but everything applies for infinite products as well.)

It is routine to verify the following:

Lemma 1.6. \mathbb{O} is an ordered commutative semi-ring with multiplicative identity $(0, 1)$ and additive identity 0.

Figure 1 shows \mathbb{O} as a topological space with the order topology.

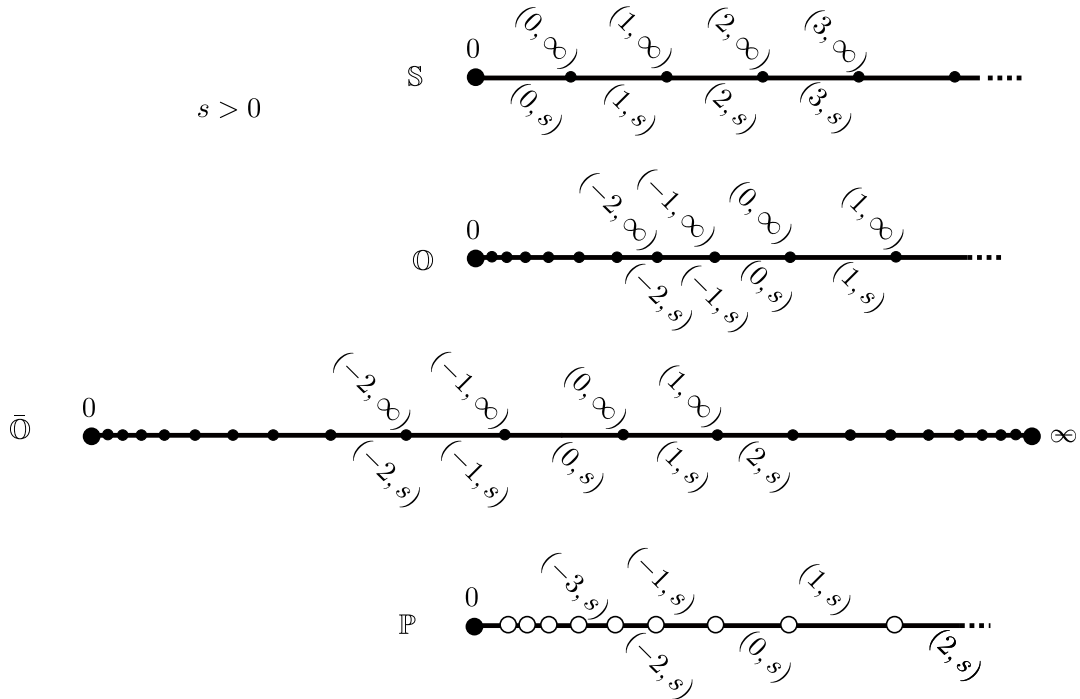


Figure 1: The ordered commutative semi-rings \mathbb{S} , \mathbb{O} , $\bar{\mathbb{O}}$, and \mathbb{P} viewed as topological spaces.

For some applications it is useful to extend \mathbb{O} slightly.

Definition 1.7. Let $\bar{\mathbb{O}}$ denote $\mathbb{O} \cup \{\infty\}$ with operations and the order in \mathbb{O} extended in the obvious way: $\infty > x$ for all $x \in \mathbb{O}$. $\infty + x = x + \infty = \infty$ for all $x \in \mathbb{O}$. Finally, $\infty \cdot x = x \cdot \infty = \infty$ unless $x = 0$, and $\infty \cdot 0 = 0 \cdot \infty = 0$.

For applications to measure theory, we will need to define countably infinite sums in \mathbb{S} , \mathbb{O} and $\bar{\mathbb{O}}$.

Definition 1.8. In \mathbb{O} we can define countably infinite sums of elements $x_n \in \mathbb{O}$, provided the summands have uniformly bounded levels. In this case, assuming the maximum of

$\mathfrak{L}(x_n)$ is M , it is natural to define $\sum_{n=1}^{\infty} x_n := \sum_{\mathfrak{L}(x_n)=M} x_n$, which makes sense since $\sum_{\mathfrak{L}(x_n)=M} x_n = (M, \sum_{\mathfrak{L}(x_n)=M} \mathfrak{R}(x_n))$, which is expressed in terms of a countable sum of positive real numbers.

If levels in $\sum_{n=1}^{\infty} x_n$ are not uniformly bounded, the countable sum does not always make sense in \mathbb{O} . However, in $\bar{\mathbb{O}}$, we can, as before, define $\sum_{n=1}^{\infty} x_n = \sum_{\mathfrak{L}(x_n)=M} x_n$ if $\{\mathfrak{L}(x_n)\}$ has maximum value M . In case $\{\mathfrak{L}(x_n)\}$ is unbounded above, we define $\sum_{n=1}^{\infty} x_n = \infty$.

In \mathbb{S} we can define $\sum_{n=1}^{\infty} x_n$, again provided the levels of summands x_n are uniformly bounded. To be able to evaluate all countable sums, we must again adjoin a new infinity, ∞ , to obtain $\bar{\mathbb{S}} = \mathbb{S} \cup \{\infty\}$.

Next, we will describe more general constructions for combining *ordered algebraic structures* which can be ordered abelian semi-groups, ordered commutative semi-rings, or ordered semi-fields, to obtain new ordered algebraic structures.

Definition 1.9. Suppose A and B are ordered abelian semi-groups. Then we define A *s-insert* B as $A \otimes B = A \times B$ which becomes an ordered semi-group, with order relation and operations described below. In the next definition we define another “insert” operation. To distinguish these, we can refer to the construction we describe here as a *semi-group insertion* or *s-insertion*.

We define an order relation $<$ on $A \otimes B$ as follows:

$(g, s) < (h, t)$ if either if $g < h$ or if $g = h$ and $s < t$.

A commutative *addition* operation on $A \otimes B$ is given by

$$(g, s) + (h, t) = \begin{cases} (g, s) & \text{if } g > h \\ (g, s + t) & \text{if } g = h \end{cases}$$

The element $(0, 0)$ is the additive identity, which we also denote simply as 0 .

If we wish to extend $A \otimes B$ by including an infinity, ∞ , then we define $A \bar{\otimes} B$ as $A \otimes B \cup \{\infty\}$ and extend the operations and order relation as follows:

For all $x \in A \otimes B$, $\infty > x$, $\infty + x = x + \infty = \infty$.

One can verify:

Lemma 1.10. *If A and B are ordered abelian semi-groups, then $A \otimes B$ and $A \bar{\otimes} B$ are ordered abelian semi-groups with the order and addition defined above.*

Definition 1.11. Suppose A is an ordered abelian group or an ordered abelian semigroup. Suppose B is an ordered abelian semi-group, an ordered commutative semi-ring, or ordered semi-field. Then we define A *insert* B as

$$A \otimes B = A \times (B \setminus \{0\}) \cup \{0\},$$

which becomes an ordered abelian semi-group or an ordered commutative semi-ring, with order relation and operations described below. The zero added to $A \oslash B$ is distinct from the zero removed from B . In case A is an ordered abelian group and B is an ordered semi-field, $A \oslash B$ becomes an ordered semi-field. In case both A and B are ordered abelian semigroups, there is a difference between s-insertion and insertion.

We define an order relation $<$ on $A \oslash B$ as follows:

- (i) $(g, s) < (h, t)$ if either if $g < h$ or if $g = h$ and $s < t$.
- (ii) $0 < (g, s)$ for all (g, s) .

Now we define the addition operation on $A \oslash B$. The commutative *addition* operation on $A \oslash B$ is given by

$$(g, s) + (h, t) = \begin{cases} (g, s) & \text{if } g > h \\ (g, s + t) & \text{if } g = h \end{cases}$$

$$0 + (g, s) = (g, s) + 0 = (g, s)$$

In case B is an ordered abelian group or ordered abelian semi-group, this defines the ordered abelian semi-group $A \oslash B$.

Provided B is an ordered commutative semi-ring, we define a commutative *multiplication* on $A \oslash B$ by

$$(g, s)(h, t) = (g + h, st),$$

$$0(g, s) = (g, s)0 = 0,$$

If we wish to extend $A \oslash B$ by including an infinity, ∞ , then we define $A \bar{\oslash} B$ as $A \oslash B \cup \{\infty\}$ and extend the operations and order relation as follows:

For all $x \in A \oslash B$, $\infty > x$, $\infty + x = x + \infty = \infty$. Finally, $\infty \cdot x = x \cdot \infty = \infty$ unless $x = 0$, and $\infty \cdot 0 = 0 \cdot \infty = 0$.

When we denote a non-zero element of $A \oslash B$ by a single symbol $x = (g, s)$, we will use $\mathfrak{L}(x) = g$ to denote the *level* of x and $\mathfrak{R}(x) = s$ to denote the *residue* of x , which lies in $B \setminus \{0\}$. We make the convention that $\mathfrak{R}(0) = 0$ and $\mathfrak{L}(0)$ is undefined (and $\mathfrak{L}(\infty)$ is undefined in $A \bar{\oslash} B$).

If A is an ordered abelian group and B is an ordered semi-field, then $A \oslash B$ is an ordered semi-field. In this case, the multiplicative inverse of (g, s) is $(-g, \bar{s})$, where \bar{s} is the multiplicative inverse of s in B .

Lemma 1.12. (a) *If A is an ordered abelian group or semi-group and B is an ordered abelian semi-group then $A \oslash B$ is an ordered abelian semi-group.*

(b) *If A is an ordered abelian group or ordered abelian semi-group and B is an ordered commutative semi-ring then $A \oslash B$ is an ordered commutative semi-ring. In this case, $\mathfrak{L}(xy) = \mathfrak{L}(x) + \mathfrak{L}(y)$. If $x \neq 0$, $y \neq 0$, then the level function \mathfrak{L} in $A \oslash B$ satisfies $\mathfrak{L}(x + y) = \max(\mathfrak{L}(x), \mathfrak{L}(y))$.*

(c) *If A is an ordered abelian group and B is an ordered commutative semi-field then $A \oslash B$ is an ordered semi-field.*

Remark 1.13. We describe the operation \otimes as an “insertion.” To construct $A \otimes B$, we insert a copy of $B \setminus \{0\}$ at every element of A , with its order, then add a different 0 to construct $A \otimes B$. To remember the meaning of the symbols \otimes and \oplus , we note that the “arrow” points down in the first, which has more limited applicability, namely it operates only on ordered abelian semi-groups.

- Examples 1.14.** (1) $\mathbb{N}_0 \otimes [0, \infty]$ is the commutative semi-ring described above as \mathbb{S} , constructed from the commutative semi-ring $[0, \infty]$ in the extended reals.
(2) $\mathbb{Z} \otimes [0, \infty]$ is the commutative semi-ring we described above as \mathbb{O} , constructed using the commutative semi-ring $[0, \infty] \subset \bar{\mathbb{R}}$.
(3) Since, for example $\mathbb{Z} \otimes [0, \infty]$ can be regarded as an ordered abelian semi-group, we can construct a new ordered commutative semi-ring $(\mathbb{Z} \otimes [0, \infty]) \otimes [0, \infty]$.
(4) We can construct many more examples by iterating as often as we want, with brackets to indicate the order of \otimes operations. A random example is $((\mathbb{Z} \otimes [0, \infty]) \otimes (\mathbb{Z} \otimes [0, \infty])) \otimes [0, \infty]$.
(5) We can first use the semi-group insertion operation to construct a new semi-group, and then construct a semi-ring using the new semi-group. For example $(\mathbb{N}_0 \otimes \mathbb{N}_0) \otimes [0, \infty]$.

In view of the examples, it is natural to ask whether in some cases the \otimes operation is associative.

Example 1.15. Consider the two ordered semi-rings $\mathbb{N}_0 \otimes (\mathbb{N}_0 \otimes \mathbb{N}_0)$ and $(\mathbb{N}_0 \otimes \mathbb{N}_0) \otimes \mathbb{N}_0$. One might hope that $\psi((i, (j, k))) = ((i, j), k)$ defines an isomorphism of ordered semi-rings. That is not the case. For example in the first group $\mathbb{N}_0 \otimes (\mathbb{N}_0 \otimes \mathbb{N}_0)$ we have $(1, (1, 1)) \cdot (2, (1, 1)) = (3, (2, 1))$ whereas in the second group $(\mathbb{N}_0 \otimes \mathbb{N}_0) \otimes \mathbb{N}_0$ we have $((1, 1), 1) \cdot ((2, 1), 1) = ((2, 1), 1)$. In fact, the two ordered semi-rings do not even have exactly the same underlying sets.

However, for semi-group insertions the situation is nicer.

Lemma 1.16. *If A, B, C are ordered abelian semi-groups, then $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.*

Proof. We can easily check that in $A \otimes (B \otimes C)$:

$$(x_1, (x_2, x_3)) + (y_1, (y_2, y_3)) = (x_1, (x_2, x_3)) \text{ if } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2,$$

$$(x_1, (x_2, x_3)) + (y_1, (y_2, y_3)) = (y_1, (y_2, y_3)) \text{ if } y_1 > x_1 \text{ or } y_1 = x_1 \text{ and } y_2 > x_2,$$

$$(x_1, (x_2, x_3)) + (y_1, (y_2, y_3)) = (x_1, (x_2, x_3 + y_3)) \text{ if } x_1 = y_1 \text{ and } x_2 = y_2.$$

Similarly, we check that in $(A \otimes B) \otimes C$:

$$((x_1, x_2), x_3) + ((y_1, y_2), y_3) = ((x_1, x_2), x_3) \text{ if } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2,$$

$$((x_1, x_2), x_3) + ((y_1, y_2), y_3) = ((y_1, y_2), y_3) \text{ if } y_1 > x_1 \text{ or } y_1 = x_1 \text{ and } y_2 > x_2,$$

$$((x_1, x_2), x_3) + ((y_1, y_2), y_3) = ((x_1, x_2), x_3 + y_3) \text{ if } x_1 = y_1 \text{ and } x_2 = y_2.$$

One can easily show that $\psi((a, (b, c))) = ((a, b), c)$ defines an order isomorphism, so the above identities prove that ψ defines an isomorphism $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. \square

Definition 1.17. If A_i are ordered abelian semi-groups, $\bigotimes_{i=1}^n A_i$ or $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ denotes the ordered abelian semi-group $((\cdots ((A_1 \otimes A_2) \otimes A_3) \otimes A_3) \cdots) A_n$.

For measure theoretic applications, it is useful to be able to evaluate all countable sums (of positive elements) in an ordered algebraic structure. To make this possible, one often has to use the insert operation $\bar{\otimes}$.

Definition 1.18. An ordered algebraic structure A has the *least upper bound property* (lub property) if every set $S \subset A$ which is bounded above has a least upper bound $\sup(S)$. The ordered algebraic structure has the *summability property* if every countable sum of positive elements can be evaluated, yielding an element in the algebraic structure.

Lemma 1.19. Suppose A is an ordered abelian group or ordered abelian semi-group with the property that every bounded non-empty set has a greatest element. Suppose B is an ordered abelian semi-group or ordered semi-ring. Suppose B has the least upper bound property and either

- (i) B has a greatest element ∞ , or
- (ii) B has a least element $p > 0$ and A has the property that every non-empty set has a least element.

Then $A \otimes B$ and/or $A \oplus B$ have the least upper bound property. Further $A \bar{\otimes} B$ and/or $A \bar{\oplus} B$ is summable.

Proof. Suppose A and B are as in the statement. We will show that $A \otimes B$ and $A \oplus B$ have the lub property. Suppose $S \subset A \otimes B$ or $S \subset A \oplus B$ is bounded above. Then

$$T = \{a \in A : \text{there exists } b \text{ such that } (a, b) \in S\}$$

is bounded above and has a greatest element M . Let $U = \{b \in B : (M, b) \in S\} \neq \emptyset$.

Case (i). If B has a greatest element ∞ , since B has the lub property and U is bounded above by ∞ , U has an lub N say. Then (M, N) is the lub for S .

Case (ii) In this case, if U is bounded above, then it has a least upper bound N , and again (M, N) is the least upper bound of S . So we now assume U is not bounded above. If A does not have a greatest element, let M' be the least element in A greater than M . Then (M', p) is the lub of S . If A has a greatest element and M is the greatest element, then for S to be bounded above, $U = \{b \in B : (M, b) \in S\} \neq \emptyset$ must also be bounded above, and (M, N) is again the lub. \square

Corollary 1.20. The ordered abelian groups $\bigotimes_{i=1}^n \mathbb{N}_0$ have the lub property. Also $\mathbb{S} = \mathbb{N}_0 \bar{\otimes} [0, \infty]$ has the lub property.

Proof. We use induction and Lemma 1.19 (ii) to show $\bigotimes_{i=1}^n \mathbb{N}_0$ has the lub property. First, we check that $\bigotimes_{i=1}^{n-1} \mathbb{N}_0$ has a least positive element, namely $(0, 0, \dots, 0, 1)$. Also, clearly \mathbb{N}_0 has the property that every bounded nonempty subset has a greatest element. Then the lemma implies $\mathbb{N}_0 \bar{\otimes} \left(\bigotimes_{i=1}^{n-1} \mathbb{N}_0 \right) = \bigotimes_{i=1}^n \mathbb{N}_0$ has the lub property.

To prove the second statement, observe that $[0, \infty]$ has a greatest element ∞ , so we can apply the lemma in case (i). \square

Lemma 1.21. *If $[0, \infty]$ denotes the interval in \mathbb{R} , then*

$$\bar{\mathbb{S}}_n = (\mathbb{N}_0 \bar{\otimes} (\mathbb{N}_0 \bar{\otimes} \cdots \bar{\otimes} (\mathbb{N}_0 \bar{\otimes} (\mathbb{N}_0 \bar{\otimes} [0, \infty])) \cdots)),$$

with n insertions of \mathbb{N}_0 , is an ordered abelian semi-ring with the lub property, is summable and has a greatest element. In particular $\bar{\mathbb{S}}$ is summable.

Similarly

$$\bar{\mathbb{O}}_n = (\mathbb{Z} \bar{\otimes} (\mathbb{Z} \bar{\otimes} \cdots \bar{\otimes} (\mathbb{Z} \bar{\otimes} (\mathbb{Z} \bar{\otimes} [0, \infty])) \cdots)),$$

with n insertions of \mathbb{Z} , is an ordered abelian semi-ring with the lub property which is also summable. In particular $\bar{\mathbb{O}}$ is summable.

Proof. We can prove the first statement using induction starting with $\mathbb{N}_0 \bar{\otimes} [0, \infty]$. We have already proved $\mathbb{N}_0 \otimes [0, \infty]$ has the lub property, and we know $\mathbb{N}_0 \bar{\otimes} [0, \infty]$ has a greatest element. Now we inductively apply $\mathbb{N}_0 \bar{\otimes}$ to the previous result, and at every step of the induction apply Lemma 1.19 (i) using $A = \mathbb{N}_0$ to prove the lub and summability properties.

Essentially the same proof works for the second statement. \square

We have described quite a few ordered algebraic structures with the desirable summability and/or lub properties using our operations \otimes , $\bar{\otimes}$, \oplus and $\bar{\oplus}$. There are more possibilities to explore, and we will do this in later sections.

2 Measures.

In this section we will assume that \mathbb{L} is some ordered abelian semigroup with at least the lub property, and also with the summability property. (When dealing with probability measures later, we can dispense with the summability property.)

Definition 2.1. Suppose \mathbb{L} is an ordered abelian semigroup with the least upper bound and summability properties. Let (X, Σ) be a measure space with σ -algebra Σ . An \mathbb{L} -measure ν assigns an element $\nu(E)$ of \mathbb{L} to each measurable set $E \in \Sigma$ such that:

- (i) $\nu(\emptyset) = 0$,
- (ii) If $\{E_i\}_{i \in I}$ is a countable collection of pairwise disjoint measurable sets in X , then

$$\nu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \nu(E_i).$$

Let us consider now the special case of an \mathbb{S} -measure. Recall $\mathbb{S} = \mathbb{N}_0 \otimes [0, \infty]$. The definitions and some ideas developed for \mathbb{S} -measures apply equally to \mathbb{O} -measures and even more generally, but for specificity for now we deal with \mathbb{S} .

Associated to an \mathbb{S} -measure ν , we will define a collection of ordinary extended-real valued measures, namely a measure ν_k associated to each level k . If $E \subset X$ is a measurable set we define

$$\nu_k(E) = \begin{cases} \Re(\nu(E)) & \text{if } \Im(\nu(E)) = k \\ \infty & \text{if } \Im(\nu(E)) > k \\ 0 & \text{if } \Im(\nu(E)) < k \text{ or if } \nu(E) = 0 \in \mathbb{S} \end{cases}$$

Notice that the measure ν can be recovered from the sequence $\{\nu_j\}$. Namely, to find $\nu(E)$ given all $\nu_i(E)$, let j be the largest i such that $\nu_i(E) > 0$, then $\nu(E) = (j, \nu_j(E)) \in \mathbb{S}$.

Observation 2.2. For measurable sets $E \subset X$ with \mathbb{S} -measure ν , $\mathfrak{L}(\nu(E))$ is uniformly bounded by $\mathfrak{L}(\nu(X))$.

The observation follows from the fact that $\nu(E) \leq \nu(X)$, so $\mathfrak{L}(\nu(E)) \leq \mathfrak{L}(\nu(X))$. But we note that if we replace \mathbb{S} by $\bar{\mathbb{S}}$ by adjoining an infinity ∞ , and $\nu(X) = \infty$, then the observation gives no useful information.

Informally, if ν is an \mathbb{S} -measure of total height h , then a set E with $\mathfrak{L}(\nu(E)) = h$ might be visible to the naked eye. To see a set with $\mathfrak{L}(\nu(E)) = h - 1$ one might need a microscope. For lower levels one would need stronger and stronger microscopes.

If one of the measures ν_j , $0 \leq j \leq h$, is trivial in the sense that there is no $E \in \Sigma$ such that $\mathfrak{L}(\nu(E)) = j$ then in some sense the finite height measure is equivalent to another one, in which there are no trivial ν_j for $0 \leq j \leq h$. In this case we could decrease by 1 the level of every ν_i for $i > j$.

Definition 2.3. Suppose ν is a finite height \mathbb{S} -measure on X . Suppose ν_j is trivial in the sense that there is no $E \in \Sigma$ such that $\mathfrak{L}(\nu(E)) = j$. Then ν is *equivalent via level alignment* to the measure μ , where μ is obtained from ν by shifting some levels: $\mu_i = \nu_i$ for $i < j$; $\mu_i = \nu_{i+1}$ for $i \geq j$. If no level alignments are possible, we say ν is a *proximal* measure.

We say the \mathbb{S} -measure ν has *total height* h if $\mathfrak{L}(\nu(X)) = h$.

More generally, suppose \mathbb{L} is an ordered abelian group which is summable with infinity ∞ and $\mathbb{K} = \mathbb{Z} \otimes \mathbb{L}$ (or $\mathbb{K} = \mathbb{N}_0 \otimes \mathbb{L}$). Again we can define \mathbb{L} -measures ν_k :

$$\nu_k(E) = \begin{cases} \mathfrak{R}(\nu(E)) & \text{if } \mathfrak{L}(\nu(E)) = k \\ \infty & \text{if } \mathfrak{L}(\nu(E)) > k \\ 0 & \text{if } \mathfrak{L}(\nu(E)) < k \text{ or if } \nu(E) = 0 \in \mathbb{K} \end{cases}$$

Many definitions we have made before carry through to $\mathbb{K} = \mathbb{Z} \otimes \mathbb{L}$ (or $\mathbb{K} = \mathbb{N}_0 \otimes \mathbb{L}$).

Definition 2.4. Suppose $\mathbb{K} = \mathbb{Z} \otimes \mathbb{L}$ (or $\mathbb{K} = \mathbb{N}_0 \otimes \mathbb{L}$). We will make definitions with notation for $\mathbb{K} = \mathbb{Z} \otimes \mathbb{L}$, but they also apply to $\mathbb{K} = \mathbb{N}_0 \otimes \mathbb{L}$.

The *total height* of a measure is *infinite* if $\mathfrak{L}(\nu(E))$ takes infinitely many values as E varies over measurable sets, otherwise it is $\max(\mathfrak{L}(\nu(E))) - \min(\mathfrak{L}(\nu(E))) + 1$.

Suppose ν is a finite height \mathbb{K} -measure on X . Suppose ν_j is trivial in the sense that there is no $E \in \Sigma$ such that $\mathfrak{L}(\nu(E)) = j$. Then ν is *equivalent via level alignment* to the measure μ , where μ is obtained from ν by shifting some levels: $\mu_i = \nu_i$ for $i < j$; $\mu_i = \nu_{i+1}$ for $i \geq j$. If no level alignments are possible, we say ν is a *proximal* measure.

The measure μ is equivalent via *level shift* to ν if for some fixed k , $\mu_i = \nu_{i-k}$.

If \mathbb{K} is commutative ordered abelian ring, we will in many cases be able to define integrals of real-valued or \mathbb{K} -valued functions with respect to \mathbb{K} -measures on measure spaces. The definition for real-valued functions will be inductive. To start the induction, we have integrals with respect to $\bar{\mathbb{R}}$ -measures.

Definition 2.5. If (X, Σ) is a measurable space, \mathbb{L} is an ordered abelian semi-ring, and $f : X \rightarrow \mathbb{L}$ is a function, we say f is *measurable* if for every $c \in \mathbb{L}$, the set $f^{-1}(\{x : x < c\})$ is measurable.

Suppose \mathbb{L} is a summable ordered abelian semi-ring with infinity ∞ , and $\mathbb{K} = \mathbb{Z}\bar{\mathbb{O}}\mathbb{L}$ (or $\mathbb{K} = \mathbb{N}_0\bar{\mathbb{O}}\mathbb{L}$). As before, we deal with $\mathbb{K} = \mathbb{Z}\bar{\mathbb{O}}\mathbb{L}$ and leave the obvious modifications for $\mathbb{K} = \mathbb{N}_0\bar{\mathbb{O}}\mathbb{L}$ to the reader. If X is a measure space with a \mathbb{K} -measure ν , and integrals of $\bar{\mathbb{R}}$ -valued measurable functions have been defined for \mathbb{L} -measures on X then we define the integral with respect to the \mathbb{K} -measure ν as follows. Let f be a real-valued measurable function on a measurable $A \subset X$. Then we define

$$\int_A f d\nu = \sum_{k \in \mathbb{Z}} (k, \int_A f d\nu_k),$$

where the sum is in \mathbb{K} . (For a finite total height measure, the above sum equals one of the summands.) Using this definition inductively, starting with $\bar{\mathbb{R}}$ -measures on X we can define the integral for ν a \mathbb{K} -measure,

$$\mathbb{K} = \bar{\mathbb{S}}_n = (\mathbb{N}_0\bar{\mathbb{O}}(\mathbb{N}_0\bar{\mathbb{O}} \cdots \bar{\mathbb{O}}(\mathbb{N}_0\bar{\mathbb{O}}(\mathbb{N}_0\bar{\mathbb{O}}[0, \infty])) \cdots)),$$

and for

$$\mathbb{K} = \bar{\mathbb{O}}_n = (\mathbb{Z}\bar{\mathbb{O}}(\mathbb{Z}\bar{\mathbb{O}} \cdots \bar{\mathbb{O}}(\mathbb{Z}\bar{\mathbb{O}}(\mathbb{Z}\bar{\mathbb{O}}[0, \infty])) \cdots)).$$

Now suppose X is a measure space with \mathbb{K} -measure ν , $\mathbb{K} = \bar{\mathbb{O}}_n$ or $\mathbb{K} = \bar{\mathbb{S}}_n$ and suppose B is a measurable set. We want to define the integral of an $\bar{\mathbb{O}}_n$ -valued function. We will deal with $\bar{\mathbb{O}}_n$, but the definition is the same for $\bar{\mathbb{S}}_n$. We have already defined the integral of a $\bar{\mathbb{R}}$ -valued measurable function with respect to ν above, so we have also defined the integral of a $\bar{\mathbb{O}}_0$ -valued function, where $\bar{\mathbb{O}}_0 = [0, \infty] \subset \bar{\mathbb{R}}$. Inductively, suppose we have defined the integral of a $\bar{\mathbb{O}}_{j-1}$ -valued measurable function on B . Let $B_k = \{x : (k-1, \infty) < g(x) \leq (k, \infty) \in \bar{\mathbb{O}}_j\}$, a measurable set. Then we define

$$\int_B g d\nu = \sum_{k \in \mathbb{Z}} \left(k, \int_{B_k} \Re(g) d\nu \right).$$

Note that $\Re(g)$ is an $\bar{\mathbb{O}}_{j-1}$ -valued measurable function on B_k which we assume is defined. We have now defined $\int_B g d\nu$ for all $\bar{\mathbb{O}}_j$ -valued functions g , $0 \leq j \leq n$.

Thus far, we have not made an issue of the “integrability” of a measurable \mathbb{K} -valued function in the above definition, $\mathbb{K} = \bar{\mathbb{O}}$; our \mathbb{K} -valued functions are positive, and the integrals may have infinite values at different levels, but otherwise measurable functions are always integrable. However, it is possible to introduce negative values for \mathbb{K} -valued functions in an artificial way, and then integrability becomes an issue.

Definition 2.6. Suppose \mathbb{L} is an ordered abelian semi-structure (without negative values). Define the *double* of \mathbb{L} as $\mathbb{DL} = \{0\} \cup \{+, -\} \times (\mathbb{L} \setminus \{0\})$. We define $y = (+, y)$, $-y = (-, y)$, $-(-, y) = y$, and $-(+, y) = (-, y) = -y$. This becomes an ordered set with the obvious ordering $(-, y) < 0$ for all $y \neq 0$, $(+, y) > 0$ for all $y \neq 0$, $(+, y) < (+, z)$ and $(-, y) > (-, z)$ if $y < z$. A function with values in \mathbb{DL} can be written as $f = f_+ - f_-$ where

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$$

and f is *measurable* if f_+ and f_- are measurable.

Suppose now that $\mathbb{L} = \mathbb{O}$ or $\mathbb{L} = \mathbb{S}$. If f is a \mathbb{DL} -valued function on an \mathbb{L} -measure space X with measure ν , and $A \subset S$ is measurable, let $P = \int_A f_+ d\nu$, $N = \int_A f_- d\nu$, the integrals of the positive and negative parts of f . We define

$$\int_A f d\nu = \begin{cases} P = \int_A f_+ d\nu & \text{if } \mathfrak{L}(P) > \mathfrak{L}(N) \\ N = -\int_A f_- d\nu & \text{if } \mathfrak{L}(P) < \mathfrak{L}(N) \\ (\mathfrak{L}(P), \mathfrak{R}(P) - \mathfrak{R}(N)) & \text{if } \mathfrak{L}(P) = \mathfrak{L}(N). \end{cases}$$

The above definition is awkward because \mathbb{DL} does not have a subtraction operation. Algebraically, \mathbb{DL} is quite defective; if one attempts to give it a structure as an additive group, the associative law fails. However, we are using an artificial sum of a negative and a positive element defined in \mathbb{DS} or \mathbb{DO} by

$$(i, s) + (-(j, t)) = \begin{cases} (i, s) & \text{if } i > j \\ -(j, t) & \text{if } i < j \\ (i, s - t) & \text{if } i = j \text{ and } s \neq t \\ 0 & \text{if } i = j \text{ and } s = t. \end{cases}$$

Example 2.7. (An interpretation of Dirac- δ functions.) We define an \mathbb{O} -measure ν for \mathbb{R}^n , determined by ν_0 and ν_{-1} . The measure ν_0 is Lebesgue measure. For ν_{-1} we use a counting measure which assigns the number of points in a set if the set is finite, and otherwise assigns ∞ . Let the Dirac- δ function δ at $y \in \mathbb{R}^n$ be defined by

$$\delta(x) = \begin{cases} (1, 1) & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_{\{y\}} \delta(x) d\nu = \int_{\mathbb{R}^n} \delta(x) d\nu = \nu(\{y\})\delta(y) = (-1, 1)(1, 1) = (0, 1)$, and for a “real-valued function” $f(x)$ with the property that $\mathfrak{L}(f(x)) = 0$, we have $\int_{\{y\}} f(x) d\nu = (-1, 1)(0, \mathfrak{R}(f(x))) = (-1, \mathfrak{R}(f(x)))$, which is trivial viewed at level 0. Also for a real-valued f and any measurable A , $\int_A f d\nu = \int_A f d\nu_0$, which is just the Lebesgue integral of f .

As with real-valued measures, if \mathbb{L} is a suitable ordered abelian semi-ring, then different \mathbb{L} -measures may be related by a Radon-Nikodym derivatives. Thus, if for all measurable E in X we have $\nu(E) = \int_E f(x) d\mu$, where f is an \mathbb{L} -valued function, we can say that f is a *Radon-Nikodym derivative* $d\nu/d\mu$.

Problem 2.8. *Formulate and prove a Radon-Nikodym theorem for \mathbb{O} -measures. It seems likely that a very similar statement is true, and that there is a simple proof involving the real measures ν_k associated to an \mathbb{O} -measure ν .*

3 Probability measures.

The author is far from expert in probability and statistics. Therefore it is again quite likely that at least some of the ideas presented in this section already exist in some form. Advice will be appreciated.

If \mathbb{L} is any ordered abelian group or ordered abelian semi-group with the lub and summability properties, \mathbb{L} -measures make sense. In order to calculate probabilities we need a division operation, so we must work with ordered semi-fields, which are much less common. We also want to avoid any kind of infinite measures, at any level, so our choices are even more limited. Besides \mathbb{R} -measures, one obvious possibility is $\mathbb{P} = \mathbb{Z} \otimes [0, \infty)$. \mathbb{P} is a sub-semi-ring of \mathbb{O} , see Figure 1. Notice that \mathbb{P} has neither the lub property nor the summability property. For fixed $i \in \mathbb{Z}$ the set $S = \{(i, t) : 0 < t < \infty\}$ is bounded above by $(i + 1, 1)$, but it does not have a least upper bound. For this reason, \mathbb{P} is also not summable. But there is also another reason that \mathbb{P} is not summable: the sum $\sum(i, 1)$ cannot be evaluated in \mathbb{P} .

Definition 3.1. If (X, Σ) is a measure space, a *probability \mathbb{P} -measure* assigns an element $\nu(E)$ of \mathbb{P} to each set $E \in \Sigma$ such that the following conditions hold:

- (i) $\nu(\emptyset) = 0$.
- (ii) For each j , if $X_j = \bigcup_{\mathfrak{L}(\nu(E))=j} E$ and $X_j \neq \emptyset$, then $\Re(\nu(X_j)) = 1$.
- (iii) If $\{E_i\}_{i \in I}$ is a countable collection of pairwise disjoint measurable sets in X , then

$$\nu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \nu(E_i).$$

Often, we assume the probability \mathbb{P} -measure is *finite depth*, meaning that the following condition also holds:

- (iv) The set $\{\mathfrak{L}(\nu(E)) : E \in \Sigma\}$ is finite.

We observe first that the set $\{\mathfrak{L}(\nu(E)) : E \in \Sigma\}$ is bounded by $\mathfrak{L}(\nu(X))$. Condition (ii) in the definition guarantees that the sum in (iii) can be evaluated and is equal to an element in \mathbb{P} .

Probability \mathbb{P} -measures are useful for calculating relative probabilities of “black swan” events. We prefer a different point of view using “depth” instead of height of levels. There is an action of \mathbb{Z} on \mathbb{O} and \mathbb{P} by multiplication. The action takes an element $x = (i, s)$ to $(k, 1)x = (k, 1)(i, s) = (k + i, s)$ for each $k \in \mathbb{Z}$, using the multiplication in \mathbb{P} . Therefore we also have an action of \mathbb{Z} on \mathbb{P} -measures so that $k \in \mathbb{Z}$ acts on ν to yield $(k, 1)\nu$.

Definition 3.2. Two \mathbb{P} -measures ν and ν' are *shift equivalent* if $\nu' = (k, 1)\nu$ for some $(k, 1) \in \mathbb{P}$, $k \in \mathbb{Z}$, which means that they are in the same orbit of the \mathbb{Z} action on \mathbb{P} -measures.

Any finite depth \mathbb{P} -measure ν is equivalent by level shift to a measure μ satisfying $-d \leq \mathfrak{L}(\nu(E)) \leq 0$ such that there exist measurable sets A and B such that $\mathfrak{L}(\nu(A)) = -d$ and $\mathfrak{L}(\nu(B)) = 0$. Using level alignment, we can further modify the measure (while possibly decreasing d) such that for each j satisfying $-d \leq j \leq 0$ there exists C such that $\mathfrak{L}(\nu(C)) = j$. When all of these conditions are satisfied, we say ν is *standard*. We also say that the measure has *total depth* d .

For a standard probability \mathbb{P} -measure, we say the probability $\nu(E)$ has *depth* j if $\mathfrak{L}(\nu(E)) = -j$.

Using the depth terminology, we can now explain probabilities of black swan events. If $\nu(E)$ has depth 0, we can imagine that $\mathfrak{R}(\nu(E))$ represents the traditional probability. If $\nu(E)$ has greater depth, it has traditional probability 0, but it has a “higher depth” non-zero probability. In this way, we assign probabilities to black swan events whose traditional probability is 0. Greater depths correspond to higher “orders of improbability.”

Example 3.3. (Dartboard example) This is an extremely simple example of a finite depth probability measure. Suppose we are given a circular dart board X with just a single cross Y drawn on it, the cross consisting of vertical and horizontal diameters. We can suppose the dart board has area 1, and suppose for simplicity that if E is Lebesgue measurable with measure $\mu(E)$, then the darts have probability $\mu(E)$ of hitting E . The traditional probability measure assigns the measure (area) of a Borel set E to the event that the dart lands on a point of E . Obviously there are many probability 0 events. For example, the probability that the dart hits the 1-dimensional cross is 0. We may define a \mathbb{P} -measure ν as follows. For any event E with positive measure in the usual sense, we define $\nu(E) = (0, e)$ where $e = \mu(E)$ is the Lebesgue measure of E . If $\mu(E)$ is 0, we define $\nu(E) = (-1, \ell)$ where ℓ is the 1-dimensional measure (length) of $E \cap Y$. We assume that the total length of Y is 1, and that each ray has length 1/4. The total depth of this \mathbb{P} -measure is 1; it has two levels. Clearly we could also define a \mathbb{P} -measure with total depth 2 (having three levels) by concentrating the depth two measure at the crossing point at the center of the dart board.

The practical usefulness of \mathbb{P} -measures comes from the fact that the usual formulas for conditional probability apply and give reasonable answers.

Definition 3.4. If A and B are events in X and ν is a probability \mathbb{P} -measure on X , then the *conditional probability of the event A given B* is $P(A|B) = \nu(A \cap B)/\nu(B)$.

Example 3.5 (Dart board example continued). Suppose B is the event that the dart hits the closed upper half of the dartboard. and Y is the event that the dart hits the cross. Then $P(Y|B) = \nu(Y \cap B)/\nu(B) = (-1, 3/4)/(0, 1/2) = (-1, 3/4)(0, 2) = (-1, 3/2)$. We obtain a probability at depth 1, but this probability is greater than $P(Y) = (-1, 1)$. This says that if the level -1 probability measure on $Y \cap B$ were “proportionately distributed” with respect to the level 0 measure on B , the the relative probability would be $(-1, 1)$, but $Y \cap B$ has 3/2 times its share of the measure, it is “over-represented” and the relative probability is $(-1, 3/2)$.

Now suppose A is the event that the dart hits the vertical ray in the upper half of the dart board. Then $P(A|Y) = (-1, 1/4)/(-1, 1) = (0, 1/4)$, in other words the conditional probability is 1/4 as one would guess. We can calculate $P(A|B \cap Y) = (-1, 1/4)/(-1, 3/4) = (0, 1/3)$, again as one would expect. Another less obvious example is $P(A|B) = (-1, 1/4)/(0, 1/2) = (-1, 1/2)$. If the measure of A were proportionately distributed in B , we would have $P(A) = (-1, 1/2)$ and $P(A|B) = (-1, 1)$, but A is under-represented in B .

Finally, Bayes’ Theorem holds in our more general probability theory.

Theorem 3.6 (Bayes Theorem). Suppose (X, Σ) is a measure space and ν is a probability \mathbb{P} -measure, which we use to calculate probabilities. Suppose $\{A_i\}$ is a partition of the event space X . Then $P(B) = \sum_j P(B|A_j)P(A_j)$ and

$$P(A_i|B) = \frac{P(B|A_i) P(A_i)}{\sum_j P(B|A_j) P(A_j)}.$$

Example 3.7 (Dart board example continued). We may suppose that the event space is the dartboard itself. Let A_1 be the closed first quadrant of the dartboard, let A_2 be the interior of the second quadrant, let A_3 be the closed third quadrant with the center removed, and A_4 is the interior of the fourth quadrant. B is the event that the dart hits the horizontal line of the cross. Now we can calculate the various probabilities in Bayes' Formula to calculate $P(A_1|B)$. For example, $P(B|A_1) = P(B \cap A_1)/P(A_1) = (-1, 1/4)/(0, 1/4) = (-1, 1)$. Similarly we calculate $P(B|A_2) = 0$, $P(B|A_3) = (-1, 1/4)$, $P(B|A_4) = 0$. Then Bayes formula gives:

$$P(A_1|B) = \frac{(-1, 1)(0, 1/4)}{(-1, 1)(0, 1/4) + (-1, 1)(0, 1/4)} = \frac{(-1, 1/4)}{(-1, 1/2)} = (0, 1/2).$$

This says the event $P(A_1|B)$ has probability $1/2$ in the usual sense.

One can use \mathbb{P} -probability distributions to produce a probability \mathbb{P} -measure starting with an arbitrary \mathbb{P} -measure on an event space X , possibly a Lebesgue measure. In particular, given a probability \mathbb{P} -measure on X , one can obtain another probability measure from a \mathbb{P} -distribution on X .

Definition 3.8. Suppose (X, μ) is a probability \mathbb{P} -measured space. Let $f : X \rightarrow \mathbb{P}$ be a function with the property that for every measurable $E \subset X$, $\nu = \int_E f d\mu \in \mathbb{P}$. In general, the integral has a value in $\bar{\mathbb{O}}$ and we are requiring that the value never be (i, ∞) or ∞ . The measure $\nu = \int_E f d\mu$ can then be made into a probability \mathbb{P} -measure by suitably normalizing the associated ν_j such that if $X_j = \bigcup_{\mathfrak{L}(\nu(E))=j} E$ and $X_j \neq \emptyset$, then $\mathfrak{R}(\nu(X_j)) = 1$

To finish this section, we ask the question whether we can define reasonable probability measures with values in other ordered algebraic structures. Some good candidates are the following:

Definition 3.9. Let

$$\mathbb{P}_n = (\mathbb{Z} \otimes (\mathbb{Z} \otimes \cdots \otimes (\mathbb{Z} \otimes (\mathbb{Z} \otimes [0, \infty)))) \cdots)$$

where \mathbb{Z} appears n times in the formula. We will denote elements of this semi-field as $(i_1, i_2, \dots, i_n, s) \in \mathbb{P}_n$, and we let $\mathfrak{L}((i_1, i_2, \dots, i_n, s)) = (i_1, i_2, \dots, i_n)$.

If (X, Σ) is a measure space a *probability \mathbb{P}_n -measure* assigns an element $\nu(E)$ of \mathbb{P}_n to each set $E \in \Sigma$ such that:

- (i) $\nu(\emptyset) = 0$,
- (ii) $\nu(E) = (i_1, i_2, \dots, i_n, s)$ satisfies $i_j \leq 0$ and $s \leq 1$.

(iii) If $\{E_i\}_{i \in I}$ is a countable collection of pairwise disjoint measurable sets in X , then

$$\nu \left(\bigcup_{i \in I} E_i \right) = \sum_{i \in I} \nu(E_i).$$

We observe that we have already built into our definition a suitable choice of representative by shift equivalence.

We can also perform alignment operations to ensure there are no “gaps,” and ν is *standard*:

(iv) If for some measurable E , $\nu(E) = (i_1, i_2, \dots, i_n, s)$ for some $s > 0$, and if $(j_1, j_2, \dots, j_n) > (i_1, i_2, \dots, i_n)$ in the lexicographical order, then there exists a measurable set E' with $\nu(E') = (j_1, j_2, \dots, j_n, t)$ for some $t > 0$.

We note first that \mathbb{P}_n is a semi-field by Lemma 1.12 (c), so the division operation makes probability calculations possible. Again, although \mathbb{P}_n does not have the lub property, condition (ii) ensures that partial sums of the sum in (iii) do have a least upper bound, so the sum makes sense.

It is difficult to imagine that even probability \mathbb{P}_2 -measures could have any practical applications, but probability \mathbb{P} -measures certainly could have applications.

4 Borel measures.

Given a Hausdorff topological space X , a Borel measure is a assigns a measure to each set in a σ -algebra generated by open sets. Usually the values of the measure lie in \mathbb{R} . Suppose \mathbb{L} is any ordered abelian semigroup with the summability property.

Definition 4.1. A Borel \mathbb{L} -measure ν on a Hausdorff topological space X assigns an element $\nu(E)$ of \mathbb{L} to each Borel set E such that:

- (i) $\nu(\emptyset) = 0$,
- (ii) If $\{E_i\}_{i \in I}$ is a countable collection of pairwise disjoint Borel sets in X , then

$$\nu \left(\bigcup_{i \in I} E_i \right) = \sum_{i \in I} \nu(E_i).$$

Example 4.2. We modify the dartboard example of the previous section. X is the unit disk in \mathbb{R}^2 , Y is the cross in X consisting of a vertical diameter union a horizontal diameter. With the usual subspace topology in \mathbb{R}^2 , and the usual measure μ on \mathbb{R}^2 , we will define an Borel \mathbb{S} -measure ν on X . It assigns $\nu(E) = (1, \mu(E)) \in \mathbb{S}$ if $\mu(E) \neq 0$, and it assigns the 1-dimensional real measure of $E \cap Y$ otherwise. More precisely, let ρ be a measure on Y agreeing with a metric which assigns length 1/4 to each ray of Y , then if $\mu(E) = 0$, we define $\nu(E) = (0, \rho(E \cap Y))$. One can verify that this is a Borel \mathbb{S} -measure. In fact, since all measures are finite at each level we have summability for measures in \mathbb{S} , so we could call this a Borel \mathbb{S} -measure.

In one way, the Borel measure in the above example is not natural, although it is natural in the setting of probability theory. Namely, there is no connection between the topology of \mathbb{S} and the topology of X . This motivates the following definition:

Definition 4.3. A Borel $\bar{\mathbb{S}}$ -measure or a Borel $\bar{\mathbb{O}}$ -measure ν on a Hausdorff topological space X is *open-graded* if the union $\bigcup E$ of Borel sets E satisfying $\nu(E) < (k, \infty)$ or $\nu(E) = 0$ is open in X .

The open-graded property is a property we will want for transverse measures on laminations.

Example 4.4. We let $X = [0, \infty]$, a subspace of the extended reals. We define an open-graded Borel $\bar{\mathbb{O}}$ -measure essentially by identifying X with our picture of $\bar{\mathbb{O}}$ in Figure 1. More specifically, choose a homeomorphism $f : X \rightarrow \bar{\mathbb{O}}$. We also have an identification $h : \mathbb{O} \rightarrow \{0\} \cup (\mathbb{Z} \times (0, \infty))$, and we let $\pi : (\mathbb{Z} \times (0, \infty)) \rightarrow (0, \infty]$ be the projection map. We let μ be Lebesgue measure on $(0, \infty]$. Given a Borel measurable set E let $\nu(E) = (j, \mu(\pi(h \circ f(E) \cap j \times (0, \infty))))$, where j is the largest i such that $\mu(\pi(h \circ f(E) \cap i \times (0, \infty)))$ is non-zero. Exceptionally, when every $\mu(\pi(h \circ f(E) \cap j \times (0, \infty)))$ is 0, we define $\nu(E) = 0$, and if every $\mu(\pi(h \circ f(E) \cap j \times (0, \infty)))$ is non-zero we let $\nu(E) = \infty$. Now it is easy to verify that ν is open-graded.

5 Laminations, metric spaces, and trees.

Suppose L is an essential codimension-1 lamination in a compact surface S , or a surface with cusps as in [3], with $\chi_g(S) < 0$. Up to minor modifications, essential laminations in a surface with $\chi_g(S) < 0$ can be realized as geodesic laminations for any chosen hyperbolic structure on S . We will usually assume that we have chosen a hyperbolic structure. The trees dual to lifts of arbitrary essential laminations are more general trees called “order trees.” There is a definition in [1], but we give a different definition. I am not sure who first defined these; I first heard about order trees from Peter Shalen.

Definition 5.1. An *order tree* is a set \mathcal{T} together with a subset $[x, y]$, called a *segment*, associated to each pair of elements, together with a linear order on $[x, y]$ such that x is the least element in $[x, y]$ and y is the greatest element. We allow *trivial segments* $[x, x]$. The set of segments should satisfy the following axioms:

- (i) The segment $[y, x]$ is the segment $[x, y]$ with the opposite order.
- (ii) The intersection of segments $[x, y]$ and $[x, z]$ is a segment $[x, w]$.
- (iii) If two segments intersect at a single point, $[x, y] \cap [y, z] = \{y\}$ then the union is a segment $[x, z]$.

The order tree is a topological space: G is open in \mathcal{T} if for every segment $[x, y]$, $G \cap [x, y]$ is open in the order topology for $[x, y]$.

Clearly \mathbb{R} -trees and $\bar{\mathbb{R}}$ -trees are also order trees, as are Λ -trees, where Λ is an ordered abelian group. Λ -trees have been studied extensively; it would be difficult to give an adequate list of references.

Suppose now that \mathbb{L} is any ordered abelian semi-group.

Definition 5.2. An \mathbb{L} -metric on a set X is a function $d : X \times X \rightarrow \mathbb{L}$ satisfying the usual axioms for a metric. An \mathbb{L} -metric space is the set X together with an \mathbb{L} -metric.

Henceforth in this section we will assume that \mathbb{L} is an ordered abelian semi-group with the summability property; we will use Borel \mathbb{L} -measures.

Definition 5.3. Suppose \mathcal{T} is an order tree and suppose ν is a Borel \mathbb{L} -measure ν on the disjoint union of segments of \mathcal{T} with the property that if $[x, y]$ and $[z, w]$ are segments, and $[x, y] \cap [z, w] = [u, v]$, then for any measurable set $E \subset [u, v]$, $\nu(E)$ is the same no matter which segment ($[x, y]$, $[z, w]$, or $[u, v]$) we use to evaluate the measure. (The measure agrees on intersections of segments.) We say ν is an \mathbb{L} -measure on \mathcal{T} , and we say \mathcal{T} with the measure ν is called an \mathbb{L} -tree.

The \mathbb{L} measure on an order tree is *non-atomic* if the measure of a single point in a segment is always 0. It has *full support* if it has full support on the disjoint union of segments.

Lemma 5.4. Suppose \mathcal{T} is an \mathbb{L} -tree with a non-atomic full support measure ν . Then \mathcal{T} is an \mathbb{L} -metric space with metric $d(x, y) = \nu([x, y]) \in \mathbb{L}$.

Proof. Because ν has no atomic measures on points, we conclude $d(x, y) = \nu([x, y]) = 0$ if and only if $x = y$. To verify the triangle inequality, observe that if x, y, z are points in the tree, by axiom (iii) for order trees, $[x, y] \cap [x, z] = [x, w]$ for some w , so $[y, w] \cup [w, z] = [y, z]$ by axiom (iii). Hence $d(y, z) = \nu([y, z]) = \nu([y, w]) + \nu([w, z]) \leq \nu([y, x]) + \nu([x, z]) = d(y, x) + d(x, z)$, because $[y, w] \subset [y, x]$ and $[w, z] \subset [x, z]$. \square

Definition 5.5. Suppose L is an essential lamination in S .

A *transverse \mathbb{L} -measure for L* is an assignment of a value $\nu(T) \in \mathbb{L}$ to every closed transversal T of the lamination. The assignment must be invariant, in the sense that an isotopy of T through transversals of L moving endpoints of the transversal in a leaf of L or in the completion of a component of $S \setminus L$ leaves $\nu(T)$ unchanged. An *\mathbb{L} -measured lamination (L, μ)* is a lamination with a transverse \mathbb{L} -measure μ .

The *order tree dual to the lift \tilde{L} of L to the universal cover \tilde{S} of S* is the set of closures of complementary regions of \tilde{L} union non-boundary leaves. A *segment $[x, y]$* is the set elements of \mathcal{T} intersected by closed oriented efficient transversal T for \tilde{L} with order coming from the order on the transversal.

If a codimension-1 lamination admits a transverse $\bar{\mathbb{S}}$ -measure or a transverse $\bar{\mathbb{O}}$ -measure which is open-graded, then the support of transverse measure ν_k is a lamination, and we have a “finite height measured lamination,” as described in [3].

Evidently a transverse \mathbb{L} -measure for a codimension-1 essential lamination in a manifold S yields a measure on the dual tree.

The following proposition was proved in a more restricted setting in [3].

Proposition 5.6. Suppose S is a compact orientable surface or a surface with cusps as in [3], with $\chi_g(S) < 0$. Given an essential lamination L in S , the associated dual tree \mathcal{T} defined above with the given segments is an order tree. If L is \mathbb{L} -measured, with measure μ , then the lifted measure $\nu = \tilde{\mu}$ yields an \mathbb{L} -measure ν for \mathcal{T} . If μ has no leaves with atomic transverse measures, \mathcal{T} is an \mathbb{L} -metric space with metric $d(x, y) = \nu([x, y])$ for $x, y \in \mathcal{T}$.

Proof. The first task is to show \mathcal{T} is an order tree. Since $\chi_g(S) < 0$, we can choose a hyperbolic structure for S .

Suppose x, y are points in \mathcal{T} , representing leaves or complementary components X and Y . A geodesic γ from a point in X to a point in Y gives an efficient transversal, hence a segment $[x, y]$ in \mathcal{T} . The uniqueness of this segment is also easy to verify: Suppose β is another efficient transversal from X to Y . Choose a geodesic segment ω in Y joining the final point of γ in Y to the final point of β in Y , and similarly choose a geodesic segment ρ joining the initial point of γ in X to the initial point of β in X . Since $\gamma\omega\bar{\beta}\bar{\rho}$ is null homotopic, we obtain a map $h : R \rightarrow \tilde{S}$ of a square R to \tilde{S} whose sides are mapped to $\gamma, \beta, \rho, \omega$. The null-homotopy h can be homotoped such that it is transverse to \tilde{L} , and can then be further homotoped such that the induced lamination on R consists of leaves joining opposite sides of R mapped to γ and β . This shows that γ and β yield the same segment in \mathcal{T} .

We verify the order tree axioms: (i) is true by construction, $[y, x]$ is $[x, y]$ with the opposite order, coming from a transversal with the opposite orientation.

For (ii), consider oriented geodesic transversal segments γ from a point in X to a point in Y , and β from a point in X to a point in Z . We may choose γ and β so that they do not intersect. Choose a geodesic segment ω joining the final point of γ to the final point of β , and similarly choose a geodesic segment ρ joining the initial point of γ in X to the initial point of β in X . The simple closed $\gamma\omega\bar{\beta}\bar{\rho}$ bounds a rectangular disk R in \tilde{S} and $\tilde{L} \cap R$ is a lamination in R which is transverse to two opposite sides γ and β with $\rho \subset \partial R$. Consider the set of leaves of $\tilde{L} \cap R$ joining γ to β . This includes at least ρ and it must be closed. So there is a largest element w in $[x, y]$ which is also in $[x, z]$.

For property (iii), suppose $[x, y]$ and $[y, z]$ are (non-trivial) segments in \mathcal{T} with $[x, y] \cap [y, z] = \{y\}$. Representing $[x, y]$ by an oriented geodesic segment β and $[y, z]$ by an oriented geodesic segment γ whose initial point is the final point of β , we see that $\beta \cup \gamma$ must be an embedded path. It follows that $\beta \cup \gamma$ can be regarded as a transversal, representing $[x, z]$.

Now that we know that \mathcal{T} is an order tree, it is easy to show it is an \mathbb{L} -tree. The transverse \mathbb{L} -measure μ for L yields a transverse measure $\tilde{\mu}$ for \tilde{L} , which in turn gives a measure on transversals. Since transversals are identified with segments of \mathcal{T} , we have a measure ν on the transversals. Invariance of the measure $\tilde{\mu}$ gives an \mathbb{L} -measure ν on the disjoint union of segments of \mathcal{T} . If there are no leaves of L with atomic measure, there are no points with atomic measure in (the segments of) \mathcal{T} , which shows that $d(x, y) = \nu([x, y])$ defines an \mathbb{L} -metric on \mathcal{T} . \square

In [3], the author investigated laminations with transverse \mathbb{S} -measures in a given surface S , defining and investigating a space of such laminations, and also investigating actions of $\pi_1(S)$ on \mathbb{S} -trees. The same program can be followed using other ordered abelian semi-groups with the summability property. In fact, the first attempt in [3] used \mathbb{O} , but it turned out that \mathbb{S} -measured laminations yielded nicer lamination spaces, though the transverse structures are quite similar. For the behavior of leaves, other transverse \mathbb{L} -structures probably encode more subtleties in the dual actions on \mathbb{L} -trees. In fact, we will give examples of laminations with “local transverse \mathbb{L} -structures”, see Section 7, which suggest that many or all laminations admit these structures. On the other hand, investigating lamination spaces for more complicated choices of \mathbb{L} may not be worthwhile.

There is a version, not included here, of Proposition 5.6 which applies to any essential

lamination in a 3-manifold with a transverse \mathbb{L} -measure. Whether we are working with codimension-1 laminations in surfaces or 3-manifolds (or even higher dimensional manifolds), it is useful to use branched manifolds with invariant \mathbb{L} -weight vectors to describe the lamination. As we have observed before in [3] (when $\mathbb{L} = \mathbb{S}$) an invariant \mathbb{L} -weight vector on a codimension-1 branched manifold embedded in a manifold does not necessarily determine an embedded \mathbb{L} -measured lamination. We will assume familiarity with the material in [3], but we recall that if a lamination with a transverse \mathbb{L} measure is carried by a branched manifold (train track or branched surface), then the transverse measure induces an \mathbb{L} weight on each sector (segment) of the branched manifold, and these weights satisfy branch equations, just as real transverse measures induce real weights on sectors satisfying branch equations. Recall also that the points in the branch locus of a train track are called *switch points*.

If B is compact with finitely many sectors, finitely many branch equations suffice to determine whether a weight vector is invariant. When $\mathbb{L} = [0, \infty) \subset \mathbb{R}$, an invariant weight vector uniquely determines a measured lamination carried by B . This is not in general true for arbitrary \mathbb{L} . A reasonable conjecture is that if $L \hookrightarrow M$ is a lamination with a transverse \mathbb{O} -measure or \mathbb{S} -measure ν , and is carried by a codimension-1 branched manifold, then the weights induced by ν on some splitting of the branched manifold B determine L and its transverse measure.

6 Mixed insertions for ordered abelian semi-groups.

We can define more general insertion operations yielding ordered abelian semi-groups. Namely, we can insert different semigroups at different levels.

Definition 6.1. Suppose A is an ordered abelian group or an ordered abelian semigroup. For each $a \in A$, let B_a be an ordered abelian semigroup. Then we define

$$A \bigotimes_{a \in A} B_a = \{(a, b) : a \in A, b \in (B_a \setminus \{0\})\} \cup \{0\}$$

We make this set into an ordered abelian semi-group by defining the order relation and the addition operation as follows:

- (i) $(g, s) < (h, t)$ if either if $g < h$ or if $g = h$ and $s < t$.
- (ii) $0 < (g, s)$ for all (g, s) .

The addition operation is commutative and given by

$$(g, s) + (h, t) = \begin{cases} (g, s) & \text{if } g > h \\ (g, s + t) & \text{if } g = h \end{cases}$$

$$0 + (g, s) = (g, s) + 0 = (g, s)$$

If we wish to extend $A \bigotimes_{a \in A} B_a$ by including an infinity, ∞ , then we define

$$A \bar{\bigotimes}_{a \in A} B_a = A \bigotimes_{a \in A} B_a \cup \{\infty\}$$

and extend the order relation and addition operation as before, so that for all $x \in A \bar{\bigotimes}_{a \in A} B_a$, $\infty > x$, $\infty + x = x + \infty = \infty$.

As before, when we denote a non-zero element of $A \bar{\bigotimes}_{a \in A} B_a$ by a single symbol $x = (g, s)$, we will use $\mathfrak{L}(x) = g$ to denote the *level of* x and $\mathfrak{R}(x) = s$ to denote the *residue of* x , which lies in $B_g \setminus \{0\}$. We make the convention that $\mathfrak{R}(0) = 0$ and $\mathfrak{L}(0)$ is undefined (and $\mathfrak{L}(\infty)$ is undefined if ∞ is included).

In the same way, we define

$$A \bigotimes_{k \leq a \leq h} B_a = \{(a, b) : a \in A, k \leq a \leq h, b \in (B_a \setminus \{0\})\} \cup \{0\}$$

which we will also write $A \bigotimes_{a=k}^h B_a$. We can similarly define $A \bigotimes_{a \leq h} B_a$ and $A \bigotimes_{k \leq a} B_a$.

Example 6.2. Let S be a surface and let \mathbb{L} be the ordered abelian group $\mathbb{N}_0 \bigotimes_{0 \leq n \leq 2} B_n$ where $B_0 = [0, \infty]$, $B_1 = [0, \infty]$ and $B_2 = \bar{\mathbb{N}}_0$. What are the essential laminations in S with transverse \mathbb{L} -measures? At the highest level, they have integer coefficients, so they are curve systems. In the complement of the curve systems, at the next level, we have a measured lamination, and in the complement of the union of the curve system with a measured lamination, we have another measured lamination. (We are ignoring the possibility of locally infinite measures, which are also allowed.) In any case, we see that \mathbb{L} -measured laminations have a very particular structure, depending on \mathbb{L} . (In this example, the \mathbb{L} -measured laminations form a subspace of the space described in [3], because \mathbb{L} is a sub semi-group of \mathbb{S} .)

7 Laminations with local transverse \mathbb{P} -structures.

There is a well-known method for constructing more interesting laminations with a given local transverse structure. Suppose we are given a lamination $L \hookrightarrow M$, where M is a manifold, and the lift \tilde{L} to the universal cover \tilde{M} has a transverse real structure μ . Further suppose that the action of $\pi_1(M)$ on \tilde{L} preserves μ , but only up to scalar multiplication. Thus there is a *stretch homomorphism* $\phi : \pi_1(M) \rightarrow \mathbb{R}_+$ such that for $\gamma \in \pi_1(M)$, the translate $\gamma(\tilde{L}, \mu) = (\tilde{L}, \phi(\gamma)\mu)$. This is called a lamination with a transverse *affine structure*, see [5], [2], [6], [7]. There are many laminations which do not have transverse \mathbb{R} -measures, but do admit affine structures. (Usually we deal with codimension-1 laminations.) The homomorphism $\log(\phi)$ can be interpreted as a cohomology class $\log \phi \in H^1(M; \mathbb{R})$.

We can play the same game with transverse \mathbb{L} structures, but we want \mathbb{L} to be a multiplicative group so that we can again define a homomorphism $\phi : \pi_1(M) \rightarrow (\mathbb{L}, \cdot)$. Thus we are forced to use $\mathbb{L} = \mathbb{P}$, or $\mathbb{L} = \mathbb{P}_n$.

Definition 7.1. A *local transverse \mathbb{P} -structure* for a lamination $L \hookrightarrow M$ is a transverse \mathbb{P} -measure μ for the lift \tilde{L} of L to the universal cover \tilde{M} of M with the property that there exists a homomorphism $\phi : \pi_1(M) \rightarrow (\mathbb{P}, \cdot)$ such that $\gamma(\tilde{L}, \mu) = (\tilde{L}, \phi(\gamma)\mu)$. We define a *local transverse \mathbb{P}_n structure* for a lamination in the same way, replacing \mathbb{P} by \mathbb{P}_n in the definition.

Regarding \mathbb{P} as a product $\mathbb{Z} \times (0, \infty)$, the homomorphism ϕ yields two homomorphisms: $\phi_1 : \pi_1(M) \rightarrow \mathbb{Z}$, which can be interpreted as an element of $H^1(M; \mathbb{Z})$ and is called a *level shift homomorphism*; and $\phi_2 : \pi_1(M) \rightarrow \mathbb{R}_+$, called the *stretch homomorphism*, where $\log \phi_2$ can be interpreted as a cohomology class in $H^1(M; \mathbb{R})$.

Examples 7.2. We will give three related examples here, two of them are laminations carried by the same branched surface in a 3-manifold. In Figure 2(a), we see a branched surface B shown immersed in \mathbb{R}^3 . Actually, it can be embedded in \mathbb{R}^3 , and we assume it is embedded and that M is a regular neighborhood of B . In the figure, we show weights on B , which are values of \mathbb{P} . We also see a transversely oriented curve α representing a cohomology class, with multiplier $(0, 1/2)$ in the transverse direction. Moving from one side of the cohomology class to the other, the weight is multiplied by $(0, 1/2)$ in \mathbb{P} . The weights and the cohomology class represent a local \mathbb{P} transverse structure on a lamination which is determined by the weights. In fact, this is an affine lamination. Since the levels of all weights and the multiplier are 0, we have real weights and a real multiplier. Cutting open the branched surface on the curve representing the cohomology class, the weights on the resulting branched surface \hat{B} represent an \mathbb{R} -measured lamination. Glueing with a stretch of $1/2$ yields the affine lamination.

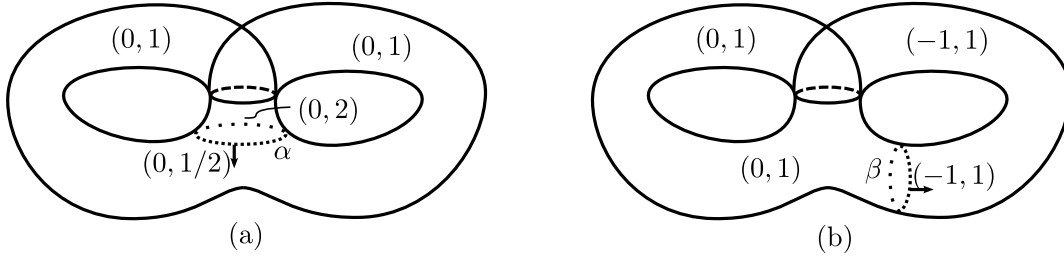


Figure 2: Two examples of laminations with local transverse \mathbb{P} structures.

Next we present an example with only a level shift homomorphism, using the same branched surface. We show weights and the cohomology class at the curve β with multiplier $(-1, 1)$, but now the multiplier at the cohomology class shifts levels. The lamination represented by the data has a “leaf spiraling on itself,” also sometimes called a spring leaf. The lamination is completely different from the one in the first example.

For a final example, we change the branched surface B as shown in Figure 3. This branched surface can also be embedded in \mathbb{R}^3 and we assume M is a regular neighborhood in \mathbb{R}^3 of B . We show a transversely oriented curve α with multiplier $(0, 1/2)$ representing the stretch homomorphism, and another transversely oriented curve β with multiplier $(-1, 1)$ representing the level shift homomorphism. Again the data determine a lamination with a local transverse \mathbb{P} structure.

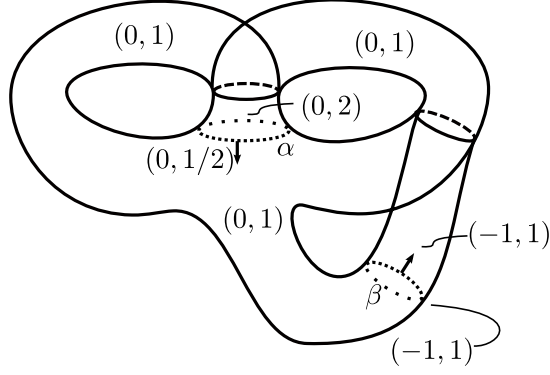


Figure 3: Example of a lamination with a local transverse \mathbb{P} structure.

We observe that in all of these examples, the data determine the lamination. That is not true in general.

Turning to local transverse \mathbb{P}_n -structures, we have the same definitions, but with n level shift homomorphisms and one stretch homomorphism.

Example 7.3. Here is an example of a lamination with a transverse local \mathbb{P}_2 -structure. In this example, we take the stretch homomorphism to be trivial, so we have two level shift homomorphisms. Again, we show a branched surface B embedded which can be embedded in \mathbb{R}^3 , and which we assume is so embedded, and we let M be a regular neighborhood of B . This is actually the same branched surface shown in Figure 3. The two level shift homomorphisms are represented by curves with multipliers in \mathbb{P}_2 , α with multiplier $(-1, 0, 1)$ and β with multiplier $(0, -1, 1)$.

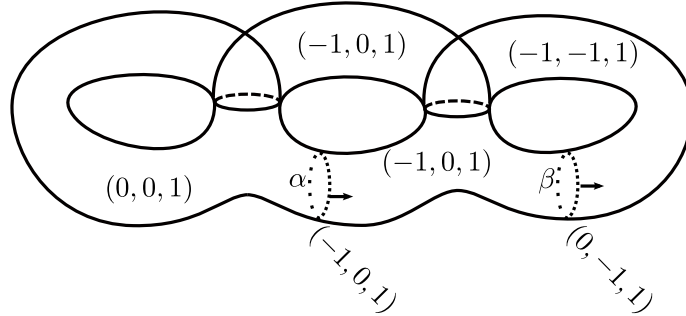


Figure 4: Example of a lamination with a local transverse \mathbb{P}_2 -structure.

Suppose L is an essential lamination in a 3-manifold M . It should be easy to describe an associated action of $\pi_1(M)$ on a \mathbb{P}_n -tree. The action does not necessarily preserve \mathbb{P}_n measures on segments; rather, it transforms measures according to stretch and level shift homomorphisms.

Is it possible that any essential 2-dimensional lamination embedded in a 3-manifold admits a full support local transverse \mathbb{P}_n -structure for some n in some regular neighborhood of itself? What about codimension-1 laminations and foliations in higher dimensional manifolds?

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